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Unified Approach to the Conservation Laws and Soliton Solution for Sine-Gordon System

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It is possible to generate an infinite number of conserved quantities and the most general soliton solution in an arbitrary background with the help of the Darboux-Backlund transformation and an expansion of the Lax eigenfunction in the eigenvalue parameter. Use is not made of the Riccati form of the Lax equation which is used in the usual derivation of the conserved quantities. It is shown that for zero seed solution one retrieves the usual one-soliton solution.

1. INTRODUCTION

Solitonlike solutions and conserved quantities of the sine-Gordon system had been obtained with the help of the inverse scattering mechanism and an analysis of the corresponding Riccati equation (Lamb, 1980). There are numerous papers on soliton solutions and their quantization (Bullough and Caudrey, 1980). There also exists the celebrated approach of the Riemann-Hilbert transform (Mikhailov, 1981; see also Zakharov *et al.*, 1984). The latter approach is able to generate a new solution by starting from an arbitrary "seed" solution, which we call the background here. In this communication we show that by using a new form of expansion for the Lax eigenfunction, it is possible to construct a formal expression for these eigenfunctions when the background solution is totally arbitrary. An interesting feature of this computation is that it also gives rise to an infinite number of conserved quantities without the use of the Riccati form of the equation. These general eigenfunctions are then used in the Darboux-Backlund transformation (Matseev, 1979*a*–*c*), leading to the most general soliton solution of the sine-Gordon system in the background of an arbitrary solution. Finally, we demonstrate that for the trivial background solution, that is $u=0$, we get back the usual one-soliton solution.

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2. FORMULATION

The sine-Gordon equation is written as

$$u_{xt} = \sin u \tag{1}$$

whence the Lax pair is

$$\psi_x = U\psi, \qquad \psi_t = V\psi \tag{2}$$

with

$$U = \begin{pmatrix} -k/2 & -u_x/2 \\ u_x/2 & k/2 \end{pmatrix} \tag{3a}$$

and

$$V = -\frac{1}{2k}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} \tag{3b}$$

Now we set the following form of the components of ψ in equation (3a):

$$\begin{aligned} \psi_1 &= c_1 \exp(-x + n_1) + c_2 h_2 \exp x \\ \psi_2 &= c_1 h_1 \exp(-x) + c_2 \exp(x + n_2) \end{aligned} \tag{4}$$

where

$$\begin{aligned} n_1 &= \sum_{n=1}^{\infty} k^{-n} \int_x^{\infty} f_n \, dx' \\ n_2 &= -\sum_{n=1}^{\infty} k^{-n} \int_{-\infty}^{x} g_n \, dx' \\ h_1 &= -\sum_{n=1}^{\infty} b_n k^{-n} \\ h_2 &= +\sum_{n=1}^{\infty} d_n k^{-n} \end{aligned} \tag{4a}$$

and c_1, c_2 are constants. After comparing different powers of k^{-m} in the two linear equations, we arrive at the following set of equations for the expansion coefficients:

$$\begin{aligned} f_{k+1} &= u_x \frac{\partial}{\partial x}\left(\frac{f_k}{u_x}\right) + \sum_{j=1}^{k-1} f_j f_{j-k} \\ g_{k+1} &= -u_x \frac{\partial}{\partial x}\left(\frac{g_k}{u_x}\right) + \sum_{j=1}^{k-1} g_j g_{j-k} \end{aligned} \tag{5}$$

with $g_1 = u_x^2/4$, $f_1 = u_x^2/4$, and $f_2 = u_x u_{xx}/4$.

The coefficients b_k and d_k are generated through

$$(b_{k+1})_x f_1 - b_{k+1} f_2 = \sum_{j=1}^{k} [b_j f_{k-j+3} - (b_j)_x f_{k-j+2}]$$

along with a similar relation for d_k.

So, given a "seed" solution u, we can determine in a recursive manner all the coefficients f_i, g_i, b_i, and d_i. Hence, the eigenfunction $(\psi_1, \psi_2)^\dagger$ is completely determined. It is interesting to note that the same computation can also be performed with equation (3b), which will determine the time evolution of these coefficients and by virtue of the basic equation (5) they should be consistent. This is indeed the case. For example, we obtain

$$\int f_{1,t}\, dx' = -\tfrac{1}{2} \cos u \tag{5'}$$

Now from our previous analysis we know that $f_1 = u_x^2/4$ and this expression really satisfies (5′) by virtue of $u_{xt} = \sin u$. The same property hold also for all f_i and g_i. Due to the above-mentioned properties, it can be ascertained that quantities like $I_k = \int_{-\infty}^{\infty} f_k\, dx$, $J_k = \int_{-\infty}^{\infty} g_k\, dx$ are conserved, due to the sine-Gordon flow. So we have obtained expressions for the conserved quantities without taking recourse to the Riccati form of the Lax pair, and our approach has the advantage that it can yield simultaneously an expression for the eigenfunction ψ given a solution u.

3. DARBOUX TRANSFORMATION

We now proceed to construct a new solution with the supposition that arbitrary u is given. For this we use the formulation of the Darboux transformation in Matveev *et al.* (1987). If such a transformation is performed N times, then the corresponding nonlinear field u is given as

$$\begin{aligned} -u_x[N] &= -ux[0] + \frac{\Delta_1(2N)}{\Delta(2N)} \\ &= -u_x[0] + \tilde{u} \quad \text{(say)} \\ \Delta(2N) &= \det(A_{ik}) \\ \Delta_1(2N) &= \det(B_{ik}) \end{aligned} \tag{6}$$

where A_{ik}, B_{ik} are defined via the following equations:

$$A_{ik} = \begin{cases} \lambda_k^{i-1} \psi_{1k}, & i = 1, 2, \ldots, N \\ \lambda_k^{i-1-N} \psi_{2k}, & i = n+1, \ldots, 2N \end{cases} \tag{7}$$

$$B_{ik} = \begin{cases} \lambda_k^{i-1} \psi_{2k}, & i = 1, 2, \ldots, N+1 \\ \lambda_k^{i-N-2} \psi_{1k}, & i = N+2, \ldots, 2N \end{cases} \tag{8}$$

$u_x[0]$ is the starting solution. Let us examine the effect of one such transformation, keeping u arbitrary. Then $\tilde{u}$ is proportional to

$$\frac{\psi_2\bar{\psi}_1}{|\psi_2|^2+|\psi_1|^2}$$

Using the expressions for ψ_1, ψ_2 given in (4), we observe

$$\tilde{u}_x \propto 4k_1 \frac{A_1 \exp(-x_1)+A_2 \exp(x_1)+A}{B_1 \exp(-x_1)+B_2 \exp(x_1)+B} \tag{9}$$

where

$$A = c_1c_2h_1\bar{h}_2 + c_1c_2 \exp(\bar{n}_1+\bar{n}_2)$$

$$B = c_1c_2(h_1\, e^{\bar{n}_2} + \bar{h}_2\, e^{n_1} + \bar{h}_1\, e^{n_2} + h_2\, e^{\bar{n}_1})$$

$$A_1 = c_1^2h_1\, e^{\bar{n}_1}, \qquad A_2 = c_2^2\bar{h}_2\, e^{n_2}$$

$$B_1 = c_1^2(h_1\bar{h}_1 + e^{n_1+\bar{n}_1})$$

$$B_2 = c_2^2(e^{n_2+\bar{n}_2} + h_2\bar{h}_2)$$

Equation (9) upon integration gives the most general form of the solution in the background of the arbitrary seed solution $u(xt)$. To prove that (9) yields the correct form in the usual case when $u=0$, let us set $u=0$ and evaluate A_i, B_i, h_i etc. We have

$$u_x \propto -2ik_1 \frac{c_1}{c_2} \frac{e^{k_1x+(4k_1/|k|^2)t}}{c_1^2/c_2^2 + e^{2[k_1x+(4k_1/|k|^2)t]}}$$

This yields upon integration

$$u \sim \tan^{-1}\left[\frac{c_2}{c_1}\exp\left(k_1x+\frac{4k_1}{|k|^2}t\right)\right]$$

which is nothing but the one-soliton solution of the sine-Gordon system.

4. DISCUSSION

In the above analysis we have shown that an expansion of the form (4) for the Lax eigenfunction can be used effectively for constructing the most general form of solution to the sine-Gordon system even when the corresponding seed (or background) solution is unknown in form. It is shown that all the coefficients in the expansions are determined recursively. The usual one-soliton solution comes out as a special case when the seed solution is zero.

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